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New integral relations for gravity waves of finite amplitude

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Some new exact relations are derived between integral properties of a progressive irrotational gravity wave of finite amplitude in deep water. In particular it is shown that the Eulerian-mean angular momentum $\bar{A}_{\mathrm{E}}$ is directly proportional to the Lagrangian $T-V$, through the relation

$$\bar{A}_{\mathrm{E}} = 2c(T-V)/g,$$

where c is the phase speed and g denotes the acceleration due to gravity. Moreover, for waves of constant length, the differential relation

$$\mathrm{d}\bar{A}_{\mathrm{E}} = 2(3T-V)\,\mathrm{d}c/g$$

also holds.

In a wave of limiting steepness it was shown previously that the level of action y_{a} is very nearly equal to the crest level $y_{\max}$. This is further discussed, and is shown to be probably a numerical coincidence.

1. Introduction

For steady irrotational gravity waves of finite amplitude, there exist a number of simple relations between the integral properties of the wave, for example the mean momentum density I, the mean densities of kinetic and potential energy T and V, and the mean fluxes of momentum and energy (see Longuet-Higgins 1975). Thus, if we take axes so that the origin is at the mean surface level and so that the Eulerian mean velocity at every point is zero, then it may be shown that, in deep water,

$$2T = cI, \tag{1.1}$$

$$F = c(3T-2V), \tag{1.2}$$

$$S_{xx} = 4T-3V, \tag{1.3}$$

where c is the phase speed, S_{xx} is the radiation stress (i.e. the excess flux of horizontal momentum due to the waves) and F is the flux of the total energy density $E = T+V$.

One useful differential relation for steady waves of fixed length, when the amplitude is allowed to vary, is

$$\mathrm{d}E = c\,\mathrm{d}I. \tag{1.4}$$

From this and (1.1) it follows that

$$\mathrm{d}L = I\,\mathrm{d}c, \tag{1.5}$$

where $L = (T-V)$ is the Lagrangian density. Equations (1.4) and (1.5) show that E and T have maxima and minima simultaneously, i.e. at the same wave steepness, and similarly for L and c, though E and c do not have simultaneous maxima in general.

Another integral quantity less commonly considered is the angular momentum density A about some fixed point (x, y). The author suggested previously (Longuet-Higgins 1980) that A may be relevant to the dynamics of wave breaking. It is easily shown that A is independent of the horizontal coordinate x, and that the a.m. density about two points at different levels y and y' are related by

$$A(y) - A(y') = (y' - y)\, I. \tag{1.6}$$

Hence $A(y) + yI$ is independent of y and is equal to $A(0)$. Unless otherwise stated we shall in the following take A to mean $A(0)$.

When considering the time-averaged value of A it is essential to distinguish between the Eulerian and the Lagrangian mean values. The Eulerian mean a.m. density $\bar{A}_{\mathrm{E}}$ is defined as the time-averaged value of A between two fixed vertical planes separated by one wavelength (in the x-direction).

The Lagrangian mean a.m. density $\bar{A}_{\mathrm{L}}$ is defined as the time average of A following the particles of fluid, and a further distinction has to be made between the short-term and the long-term averages $\bar{A}_{\mathrm{L}}$ (see Longuet-Higgins 1980, §8). Here, however, we shall be considering in detail only the Eulerian-mean density $\bar{A}_{\mathrm{E}}$, and we shall prove, among other results, the very simple relation

$$\bar{A}_{\mathrm{E}} = 2c(T - V)/g \tag{1.7}$$

connecting the Eulerian mean angular momentum with the Lagrangian density $L = T - V$. It follows immediately that $\bar{A}_{\mathrm{E}}$ is stationary at the same wave amplitudes as are L and c.

Finally in §9 we consider the Lagrangian-mean angular momentum, and show that a near equality between the crest height of a wave of limiting steepness and the corresponding level of action (suitably defined) is probably a numerical coincidence.

2. Definitions

Consider a steady progressive wave, of wavelength λ, travelling in the x-direction with speed c (see figure 1). The y-axis being vertically upwards, the free surface is given by $y = \eta$, a function of $x - ct$, and we may choose the origin so that

$$\bar{\eta} \equiv \frac{1}{\lambda}\int_0^\lambda \eta \,\mathrm{d}x = 0. \tag{2.1}$$

Further, if (u, v) denote the components of the velocity (assumed irrotational), we may choose axes so that

$$\bar{u} \equiv \frac{1}{\lambda}\int_0^\lambda u \,\mathrm{d}x = 0 \tag{2.2}$$

at any fixed point below the wave troughs. The mean densities of horizontal momentum (or impulse) and of kinetic and potential energy are defined by

$$I = \overline{\int_{-h}^{\eta} u \,\mathrm{d}y}, \tag{2.3}$$

$$T = \overline{\int_{-h}^{\eta} \tfrac{1}{2}(u^2 + v^2)\,\mathrm{d}y}, \tag{2.4}$$

$$V = \overline{\int_0^\eta gy \,\mathrm{d}y}, \tag{2.5}$$

where h is the mean depth (possibly infinite) and an overbar denotes the mean value with respect to x or t.

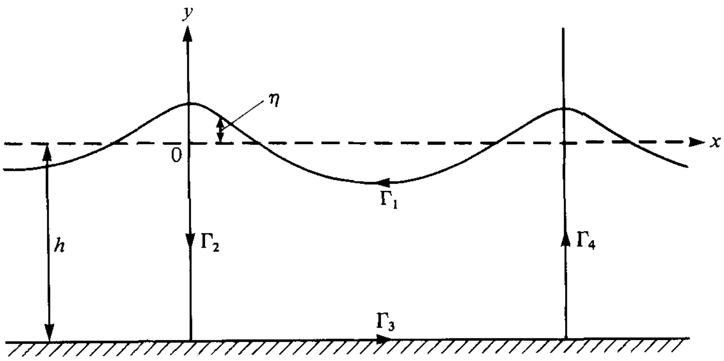

FIGURE 1. Sketch of a progressive wave, showing the contour of integration Γ.

The angular momentum density A_E, after § 1, is defined by

$$\bar{A}_E = \overline{\frac{1}{\lambda}\int_0^\lambda \int_{-h}^{\eta} (yu - xv)\,dy\,dx}, \tag{2.6}$$

where a bar denotes the time average. In this expression, since the limits of x are fixed, we may reverse the order of time averaging and of integration with respect to x. But because the wave is progressive,

$$\overline{\int_{-h}^{\eta} v\,dy} = \frac{1}{\lambda}\int_0^\lambda \int_{-h}^{\eta} v\,dy\,dx, \tag{2.7}$$

which must vanish, since the total vertical momentum in the wave is zero. Hence the contribution of the term xv to the integral (2.6) is zero, and we have simply

$$\bar{A}_E = \overline{\int_{-h}^{\eta} yu\,dy}. \tag{2.8}$$

3. Transformations

Let ϕ denote the velocity potential, so that $(u, v) = (\phi_x, \phi_y)$, and from (2.2)

$$[\phi]_{x=0}^{\lambda} = 0, \tag{3.1}$$

that is, ϕ is periodic in x. Then from (2.3) we have

$$\lambda I = \int_0^\lambda \int_{-h}^{\eta} \phi_x\,dy\,dx = \int_\Gamma \phi\,dy \tag{3.2}$$

where Γ is the contour shown in figure 1. The contributions from the two vertical sides Γ_2 and Γ_4 cancel, by (3.1). The contribution from the horizontal bottom Γ_3 vanishes since $dy = 0$. We are left with the contribution from the free surface Γ_1, and on integrating by parts we have

$$\lambda I = [\phi y]_{x=0}^{\lambda} - \int_{\Gamma_1} y\,d\phi. \tag{3.3}$$

The first term on the right again vanishes by periodicity, hence

$$I = \frac{1}{\lambda}\int_{x=0}^{\lambda} \eta\,d\phi \tag{3.4}$$

(for an alternative proof see Longuet-Higgins 1975, §2).

Similarly from (2.8) we have by Green's theorem

$$\lambda \bar{A}_{\mathrm{E}} = \int_0^\lambda \int_{-h}^{\eta} y\phi_x \, \mathrm{d}y \, \mathrm{d}x = \int_\Gamma y\phi \, \mathrm{d}y, \tag{3.5}$$

so

$$\bar{A}_{\mathrm{E}} = \frac{1}{\lambda} \int_{\Gamma_1} y\phi \, \mathrm{d}y, \tag{3.6}$$

and after integration by parts

$$\bar{A}_{\mathrm{E}} = \frac{1}{\lambda} \int_{x=0}^{\lambda} \tfrac{1}{2} y^2 \, \mathrm{d}\phi. \tag{3.7}$$

In addition we have from (1.1), (3.2) and (3.4) two expressions for the kinetic energy, namely

$$T = \frac{1}{\lambda} \int_{\Gamma_1} \tfrac{1}{2} c\phi \, \mathrm{d}y \tag{3.8}$$

and

$$T = \frac{1}{\lambda} \int_{x=0}^{\lambda} \tfrac{1}{2} c\eta \, \mathrm{d}\phi. \tag{3.9}$$

4. Fourier series

Consider waves in deep water, where $\phi \to 0$ as $y \to -\infty$, and let Φ denote the velocity potential in the steady flow as seen by an observer moving horizontally with the phase velocity c. Clearly

$$\Phi = \phi - cX, \quad \text{where} \quad X = x - ct. \tag{4.1}$$

Let Ψ denote the corresponding stream function. Then X and y are conjugate functions of Φ and Ψ, and, if units are chosen so that

$$\lambda = 2\pi, \qquad k \equiv \frac{2\pi}{\lambda} = 1, \tag{4.2}$$

we may then write

$$\left.\begin{aligned} X &= -\frac{\Phi}{c} - \sum_{n=1}^{\infty} a_n \, \mathrm{e}^{-n\Psi/c} \sin \frac{n\Phi}{c}, \\ y + K &= -\frac{\Psi}{c} + \sum_{n=1}^{\infty} a_n \, \mathrm{e}^{-n\Psi/c} \cos \frac{n\Phi}{c}, \end{aligned}\right\} \tag{4.3}$$

where a_n and K are real constants. Taking $\Psi = 0$ on the free surface, we then have

$$\left.\begin{aligned} X &= -\frac{\Phi}{c} - \sum_1^{\infty} a_n \sin \frac{n\Phi}{c}, \\ \eta &= -K + \sum_1^{\infty} a_n \cos \frac{n\Phi}{c}. \end{aligned}\right\} \tag{4.4}$$

The condition $\bar{\eta} = 0$ yields

$$0 = \frac{1}{\lambda} \int_0^\lambda \eta \, \mathrm{d}X = K - \frac{1}{2} \sum_1^{\infty} n a_n^2, \tag{4.5}$$

so that

$$K = \frac{1}{2} \sum_1^{\infty} n a_n^2. \tag{4.6}$$

From (3.4) we have also

$$I = \frac{1}{\lambda}\int_{X=0}^{\lambda} \eta\,\mathrm{d}(\Phi + cX) = \frac{1}{\lambda}\int_{X=0}^{\lambda} \eta\,\mathrm{d}\Phi, \tag{4.7}$$

since $\bar{\eta} = 0$. So from (4.4)

$$I = cK. \tag{4.8}$$

From (1.1) we then have

$$2T = c^2 K, \tag{4.9}$$

where K is given by (4.6). Analogous expressions for V and $\bar{A}_{\mathrm{E}}$ will be derived in §6.

5. Proof of (1.7)

Let us write $Y = y - c^2/2g$ and consider the integral

$$F(n, \Psi) = \int_{X=0}^{2\pi} \frac{\mathrm{e}^{-\mathrm{i}n\Phi/c}}{X_\Phi - \mathrm{i}\,Y_\Phi}\,\mathrm{d}\Phi, \tag{5.1}$$

where n is any integer and the integral is taken along any streamline Ψ = constant. By differentiating (5.1) with respect to Ψ and using the Cauchy–Riemann relations $X_\Psi = -\,Y_\Phi$, $Y_\Psi = X_\Phi$, it can easily be shown that

$$\frac{\partial F}{\partial \Psi} = \frac{n}{c}F \tag{5.2}$$

(see Longuet-Higgins 1978). Therefore

$$F = F_n\,\mathrm{e}^{n\Psi/c}, \tag{5.3}$$

where F_n is a constant. But as $Y \to -\infty$ so $\Psi \to \infty$ and $X_\Phi - \mathrm{i}Y_\Phi \to -1/c$, which is bounded. So when $n > 0$ we must have $F_n = 0$, and the integral vanishes identically; while for $n = 0$, F is a constant, equal to its limit as $\Psi \to \infty$, that is $2\pi c^2$.† Since

$$\frac{1}{X_\Phi - \mathrm{i}\,Y_\Phi} = \frac{X_\Phi + \mathrm{i}\,Y_\Phi}{X_\Phi^2 + Y_\Phi^2} = q^2(X_\Phi + \mathrm{i}Y_\Phi), \tag{5.4}$$

where

$$q^2 = \Phi_X^2 + \Phi_Y^2 = (u-c)^2 + v^2, \tag{5.5}$$

this shows that

$$\frac{1}{\lambda}\int_{X=0}^{\lambda} q^2\,\mathrm{e}^{-\mathrm{i}n\Phi/c}(\mathrm{d}X + \mathrm{i}\,\mathrm{d}Y) = \begin{cases} c^2 & (n = 0), \\ 0 & (n = 1, 2, \ldots). \end{cases} \tag{5.6}$$

Now at the free surface we have from Bernoulli's equation, if we take $g = 1$,

$$y + \tfrac{1}{2}q^2 = \tfrac{1}{2}c^2, \quad Y + \tfrac{1}{2}q^2 = 0. \tag{5.7}$$

On substituting for q^2 in (5.6) we obtain

$$\frac{1}{\lambda}\int Y\,\mathrm{e}^{-\mathrm{i}n\Phi/c}(\mathrm{d}X + \mathrm{i}\,\mathrm{d}Y) = \begin{cases} -\tfrac{1}{2}c^2 & (n = 0), \\ 0 & (n = 1, 2, \ldots). \end{cases} \tag{5.8}$$

But from (4.4)

$$Y + \mathrm{i}X + \frac{\mathrm{i}\Phi}{c} = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\,\mathrm{e}^{-\mathrm{i}n\Phi/c}, \tag{5.9}$$

† Because Φ runs from 0 to $-2\pi c$, $\mathrm{d}\Phi$ is negative.

where
$$a_0 = -2K - c^2. \tag{5.10}$$
Now multiply each side of (5.9) by $\lambda^{-1} Y(\mathrm{d}X + \mathrm{i}\,\mathrm{d}Y)$ and integrate term-by-term using (5.8). We get
$$\frac{1}{\lambda}\int\left(Y + \mathrm{i}X + \frac{\mathrm{i}\Phi}{c}\right) Y(\mathrm{d}X + \mathrm{i}\,\mathrm{d}Y) = -\tfrac{1}{4}c^2 a_0. \tag{5.11}$$
The real part gives us
$$\frac{1}{\lambda}\int Y^2\,\mathrm{d}X - \frac{1}{\lambda}\int\left(X + \frac{\Phi}{c}\right) Y\,\mathrm{d}Y = -\tfrac{1}{4}c^2 a_0. \tag{5.12}$$
The first integral in (5.12) equals
$$\frac{1}{\lambda}\int (\eta - \tfrac{1}{2}c^2)^2\,\mathrm{d}X = 2V + \tfrac{1}{4}c^4, \tag{5.13}$$
since $\bar{\eta} = 0$ by (2.1). The second integral in (5.12) equals
$$-\frac{1}{\lambda}\int \frac{\phi}{c}(\eta - \tfrac{1}{2}c^2)\,\mathrm{d}y = \frac{\bar{A}_{\mathrm{E}}}{c} - T, \tag{5.14}$$
by (3.6) and (3.8). Thus altogether (5.12) gives
$$(2V + \tfrac{1}{4}c^4) + \left(\frac{\bar{A}_{\mathrm{E}}}{c} - T\right) = -\tfrac{1}{4}c^2(2K + c^2), \tag{5.15}$$
that is
$$\bar{A}_{\mathrm{E}}/c = T - 2V + \tfrac{1}{2}c^2 K \tag{5.16}$$
$$= 2T - 2V, \tag{5.17}$$
by (4.9). This proves (1.7) when $g = 1$, and hence generally.

6. Further relations for V and $\bar{A}_{\mathrm{E}}$

In §4 we found expressions for I and T in terms of the Fourier coefficients a_n, through the relation (4.6) for K. We shall now find corresponding expressions for V and $\bar{A}_{\mathrm{E}}$.

Let us define
$$J = \frac{1}{2}\sum_1^\infty a_n^2. \tag{6.1}$$
Then it will be shown that
$$6V = J + c^2 K + K^2 \tag{6.2}$$
and
$$6\bar{A}_{\mathrm{E}}/c = -J + c^2 K - K^2 \tag{6.3}$$
(g being equal to 1).

Starting from (3.7), we have
$$\bar{A}_{\mathrm{E}} = \frac{1}{\lambda}\int_{X=0}^{\lambda} \tfrac{1}{2}\eta^2\,\mathrm{d}(\Phi + cX), \tag{6.4}$$
by (4.1). On substituting for η the series in (4.4), we have
$$\frac{1}{\lambda}\int_{X=0}^{\lambda} \tfrac{1}{2}\eta^2\,\mathrm{d}\Phi = -\tfrac{1}{2}c(J + K^2), \tag{6.5}$$

and from (2.5)
$$\frac{1}{\lambda}\int \tfrac{1}{2}\eta^2\,\mathrm{d}X = V. \tag{6.6}$$

So
$$2\bar{A}_{\mathrm{E}} = c(2V - J - K^2). \tag{6.7}$$

But from (5.16) and (4.10) we have also
$$\bar{A}_{\mathrm{E}} = c(c^2K - 2V). \tag{6.8}$$

On eliminating $\bar{A}_{\mathrm{E}}$ and V in turn from these last two equations, we obtain (6.2) and (6.3).

We can also write implicit expressions for the total energy density $E = T+V$ and the Lagrangian density $L = T-V$; thus
$$6E = J + 5c^2K + K^2 \tag{6.9}$$

and
$$6L = -J + c^2K - K^2. \tag{6.10}$$

Through (5.10) we can also eliminate c^2 and express V, $\bar{A}_{\mathrm{E}}$, E and L in terms of a_0 and $a_1, a_2, \ldots$.

For waves of small amplitude a_1 we have
$$J \sim K \sim \tfrac{1}{2}a_1^2, \qquad c^2 \sim 1 + a_1^2, \tag{6.11}$$

and it is seen clearly from (6.9) and (6.10) that
$$E \sim \tfrac{1}{2}a_1^2, \qquad L \sim \tfrac{1}{8}a_1^4, \tag{6.12}$$

while $\bar{A}_{\mathrm{E}}$, like L, is $O(a_1^4)$, as shown in Longuet-Higgins (1980, §6).

7. Relations between the a_n

The set of integral relations (5.8) is easily shown to be equivalent to a set of quadratic relations between the coefficients a_n in the Fourier series (5.9) for η (see Longuet-Higgins 1978; previously the simplest known relations were cubic). These quadratic relations may be stated in the form
$$\left.\begin{aligned}
&a_0 + a_1a_1 + 2a_2a_2 + 3a_3a_3 + \ldots = -c^2,\\
&a_1 + a_0a_1 + 2a_1a_2 + 3a_2a_3 + \ldots = 0,\\
&a_2 + a_1a_1 + 2a_0a_2 + 3a_1a_3 + \ldots = 0,\\
&a_3 + a_2a_1 + 2a_1a_2 + 3a_0a_3 + \ldots = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{7.1}$$

The first of these gives
$$a_0 + 2K = -c^2, \tag{7.2}$$

as in (5.10). It is the only one involving c^2 explicitly. The remaining equations may be recast in the form
$$\left.\begin{aligned}
&\phantom{a_1a_1+{}}2a_1a_2 + 3a_2a_3 + 4a_3a_4 + \ldots = -a_1 - a_0a_1,\\
&a_1a_1 \phantom{{}+2a_1a_2} + 3a_1a_3 + 4a_2a_4 + \ldots = -a_2 - 2a_0a_2,\\
&a_2a_1 + 2a_1a_2 \phantom{{}+3a_1a_3} + 4a_1a_4 + \ldots = -a_3 - 3a_0a_3,\\
&a_3a_1 + 2a_2a_2 + 3a_1a_3 \phantom{{}+4a_1a_4} + \ldots = -a_4 - 4a_0a_4,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{7.3}$$

Let us multiply the first of (7.3) by a_1, the second by a_2, and so on, and add. If α and β denote the sum of terms below and above the diagonal respectively, then it is easily seen that

$$\begin{aligned} 2\alpha &= a_2(2a_1 a_1) \\ &\quad + a_3(3a_2 a_1 + 3a_1 a_2) \\ &\quad + a_4(4a_3 a_1 + 4a_2 a_2 + 4a_1 a_3) \\ &\quad + \dots \\ &= \beta. \end{aligned} \tag{7.4}$$

But summing the right-hand sides of (7.3) we have

$$\alpha + \beta = -2J - 2a_0 K. \tag{7.5}$$

Therefore
$$2\alpha = \beta = -\tfrac{4}{3}(J + a_0 K). \tag{7.6}$$

Now if in
$$2V = \frac{1}{\lambda}\int_0^\lambda \eta^2 \,\mathrm{d}X \tag{7.7}$$

we substitute the series (4.4) for X and η we obtain

$$2V = J - K^2 + R, \tag{7.8}$$

where

$$\begin{aligned} R &= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_1^\infty a_n \cos\frac{n\Phi}{c}\right]^2 \left[\sum_1^\infty m a_m \cos\frac{m\Phi}{c}\right] \mathrm{d}\left(\frac{\Phi}{c}\right) \\ &= \tfrac{1}{2}\alpha + \tfrac{1}{4}\beta & (7.9) \\ &= -\tfrac{2}{3}(J + a_0 K), & (7.10) \end{aligned}$$

by (7.5). On substituting for a_0 from (7.2), we find that (7.8) reduces to (6.2).

8. Differential relations for $\bar{A}_\mathrm{E}$

From the basic result

$$\bar{A}_\mathrm{E} = 2cL \tag{8.1}$$

(when $g = 1$) it follows that

$$\mathrm{d}\bar{A}_\mathrm{E} = 2(L\,\mathrm{d}c + c\,\mathrm{d}L), \tag{8.2}$$

or from (1.5)
$$\mathrm{d}\bar{A}_\mathrm{E} = 2(L + cI)\,\mathrm{d}c, \tag{8.3}$$

that is
$$\mathrm{d}\bar{A}_\mathrm{E} = 2(3T - V)\,\mathrm{d}c. \tag{8.4}$$

This shows that $\bar{A}_\mathrm{E}$ has maxima or minima at the same wave amplitudes as does the phase speed c and the Lagrangian L.

As regards the first maximum of c, this conclusion may be verified from the numerical values of $\bar{A}_\mathrm{E}$, c, T and V given in tables 1 and 2 of Longuet-Higgins (1980).

9. The Lagrangian-mean density $\bar{A}_L$

Exact relations for the Lagrangian-mean angular momentum $\bar{A}_L$ are more difficult to come by. For reference, however, it may be useful to state concisely the analytical results derived by Longuet-Higgins (1980) for the long-time Lagrangian-mean, l.t. $\bar{A}_L$.

If we define

$$S(\Psi) = \frac{1}{2\pi}\int_0^{2\pi}\left(y+K+\frac{\Psi}{c}\right)\left(\frac{c}{q}\right)^2 \mathrm{d}\left(\frac{\Phi}{c}\right) \tag{9.1}$$

and

$$S'(\Psi) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{c}{q}\right)^2 \mathrm{d}\left(\frac{\Phi}{c}\right), \tag{9.2}$$

then it was shown that (in the present notation)

$$\text{l.t. } \bar{A}_L = c(V-J-\tfrac{1}{2}K^2)+cQ, \tag{9.3}$$

where

$$cQ = \int_0^{\infty}\frac{S}{S'}\mathrm{d}\Psi. \tag{9.4}$$

In (9.3) $\bar{A}_L$ signifies the Lagrangian-mean angular momentum $\overline{A_L(0)}$ about the level $y = 0$. By (1.1), the mean angular momentum $\overline{A_L(y)}$ about any other level is given by

$$\overline{A_L(y)} = A_L(0) - yI. \tag{9.5}$$

We may then define the *level of action* y_a as the height above the mean level for which l.t. $\overline{A_L(y)}$ vanishes, that is

$$y_a = \text{l.t. } \overline{A_L(0)}/I. \tag{9.6}$$

The numerical calculations given in Longuet-Higgins (1980) showed that as the wave steepness ak increases from 0 to its limiting value for a steady uniform wave, $ak \sim 0.4434$, so y_a increases from 0.5 k^{-1} to about 0.59 k^{-1}, quite close to the crest height of a limiting wave, namely 0.596 k^{-1}. Despite further calculations by Williams (1981), it is not yet known whether the two heights are theoretically equal. The problem presents an interesting challenge.

Previously, the integral in (9.4) was evaluated numerically using the series expansions of S and S':

$$S = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} a_m b_n b_{m+n}\,\mathrm{e}^{-2(m+n)\Psi/c}, \tag{9.7}$$

$$S' = \sum_{n=0}^{\infty} b_n^2\,\mathrm{e}^{-2n\Psi/c}, \tag{9.8}$$

where

$$b_0 = 1, \qquad b_n = na_n \quad (n = 1, 2, \ldots). \tag{9.9}$$

Here we shall take the analysis a step further. We write

$$S/S' = S'', \tag{9.10}$$

where

$$S''(\Psi) = \sum_{m=1}^{\infty} c_m\,\mathrm{e}^{-2m\Psi/c}. \tag{9.11}$$

The coefficients c_m may be determined by comparing coefficients of $\mathrm{e}^{-2(m+n)\Psi/c}$ in the relation

$$S'S'' = S. \tag{9.12}$$

This gives

$$\sum_{m+n=l} b_n^2 c_n = \sum_{m+n=l} a_m b_n b_l \quad (l = 1, 2, \ldots), \tag{9.13}$$

that is

$$\left.\begin{array}{ll} 1\cdot c_1 & = (a_1) b_1, \\ b_1^2 c_1 + 1\cdot c_2 & = (a_2 + a_1 b_1) b_2, \\ b_2^2 c_1 + b_1^2 c_2 + 1\cdot c_3 & = (a_3 + a_2 b_1 + a_1 b_2) b_3, \\ b_3^2 c_1 + b_2^2 c_2 + b_1^2 c_3 + 1\cdot c_4 & = (a_4 + a_3 b_1 + a_2 b_2 + a_1 b_3) b_4, \\ \ldots\ldots\ldots\ldots & \end{array}\right\} \tag{9.14}$$

From these equations $c_1, c_2, \ldots$ may be determined in succession. To determine Q, however, we need only find, from (9.4),

$$Q = \int_0^\infty S'' \,\mathrm{d}\left(\frac{\Psi}{c}\right) = \frac{1}{2}\sum_1^\infty m^{-1} c_m. \tag{9.15}$$

From (9.15), this can be expressed in the form of a determinant

$$Q = \begin{vmatrix} 0 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4}\ldots \\ a_1 b_1 & 1 & 0 & 0 & 0\ldots \\ (a_2 + a_1 b_1) b_2 & b_1^2 & 1 & 0 & 0\ldots \\ (a_3 + a_2 b_1 + a_1 b_2) b_3 & b_2^2 & b_1^2 & 1 & 0\ldots \\ (a_4 + a_3 b_1 + a_2 b_2 + a_1 b_3) b_4 & b_3^2 & b_2^2 & b_1^2 & 1\ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix} \tag{9.16}$$

in which all the elements are known combinations of the Fourier coefficients.

We are interested in the value of Q subject to the condition for limiting waves, that $q = 0$ at the wave crest. From (5.7) this implies that $Y_{\Phi=0} = 0$, and so from (5.9)

$$\sum_1^\infty a_n = -\tfrac{1}{2} a_0. \tag{9.17}$$

Now the crest height of a limiting wave above the mean surface level $y = 0$ is given by setting $q = 0$ in (5.7), that is to say

$$y_{\max} = \tfrac{1}{2}c^2. \tag{9.18}$$

So the conjecture that for the limiting wave $y_{\mathrm{a}} = y_{\max}$ is equivalent to the statement that, subject to the condition (9.17),

$$c(V - J - \tfrac{1}{2}K^2) + cQ = \tfrac{1}{2}c^2 I. \tag{9.19}$$

Since $I = cK$ this can be written

$$Q = (J + \tfrac{1}{2}c^2 K + \tfrac{1}{2}K^2) - V, \tag{9.20}$$

or by (6.2)

$$Q = \tfrac{1}{6}(5J + c^2 K + 2K^2). \tag{9.21}$$

In view of (7.2) this reduces simply to

$$Q = \tfrac{1}{6}(5J - a_0 K). \tag{9.22}$$

It is not clear that the two expressions (9.16) and (9.22) are equal, even when account is taken of (7.3) and (9.17) and the definitions of J and K. Thus it appears that the closeness of y_{a} to $y_{\max}$ for the highest wave is a purely numerical coincidence.

The physical consequences of this coincidence, discussed in Longuet-Higgins (1980, §10), remain unaffected.

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